SPECIAL LINEAR SYSTEMS ON TORIC VARIETIES

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ABSTRACT. We consider linear systems on toric varieties of any dimension, with invariant base points, giving a characterization of special linear systems. We then make a new conjecture for linear systems on rational surfaces.

INTRODUCTION

Let us take the projective plane $\mathbb{P}^2$ and let us consider the linear system of curves of degree d having some points of fixed multiplicity. The expected dimension of such systems is the dimension of the space of degree d polynomials minus the conditions imposed by the multiple points. The systems whose dimension is bigger than the expected one are called *special systems*.
There exists a conjecture due to Hirschowitz (see [3]), characterizing special linear systems on $\mathbb{P}^2$, which has been proved in some special case [1, 2, 6, 5] There exists also an analogous conjecture for rational ruled surfaces [4].
Concerning the dimension of linear systems on varieties of bigger dimension, with some points of fixed multiplicity, very few is known.
In this article we study this problem in the case of toric varieties, considering only points which are invariant under the action of the maximal torus. In particular we take a smooth n-dimensional toric variety X, an ample divisor D on it, r equivariant points $p_1, p_2, \ldots, p_r$ and r non negative integers $m_1, m_2, \ldots, m_r$. We denote by $\mathcal{L}(D, m_1, \ldots, m_r)$ the linear system of the divisors linearly equivalent to D and passing through p_i with fixed multiplicity m_i.
Then we prove the following:

Main Theorem. *The linear system $\mathcal{L}(D, m_1, \ldots, m_r)$ is special if and only if there exists an equivariant curve C passing through two of the p_i's, such that $\mathcal{L}(D, m_1, \ldots, m_r) \cdot C \leq -2$.*

The paper is organized as follows: in Section 1 we recall some definitions and we fix some notations (see for instance [7] for a complete reference), while in Section 2 we state and prove our main theorem and we formulate a new conjecture for linear systems on rational surfaces.

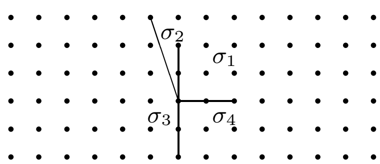
1. DEFINITIONS AND NOTATIONS

In what follows we will adopt the notation of [7], Chapter 2.
Let $N \cong \mathbb{Z}^n$ and let M be its dual $\mathbb{Z}$-module; we denote by $N_{\mathbb{R}}$ (resp. $M_{\mathbb{R}}$) the $\mathbb{R}$-module $N \otimes \mathbb{R}$ (resp. $M \otimes \mathbb{R}$). Given a fan Δ in N, the corresponding n-dimensional toric variety X is denoted by $T_N emb(\Delta)$. In this paper we consider only smooth toric varieties defined by nonsingular, complete fans. We denote by $\sigma_1, \ldots, \sigma_k$ the n-dimensional cones of Δ corresponding to the T_N-invariant points of X, $p_1, \ldots, p_k$.

Given an ample divisor D on X, let $h \in SF(N,\Delta)$ be its Δ-linear support function. The convex polytope in $M_{\mathbb{R}}$ representing $H^0(X,\mathcal{O}_X(D))$ is denoted by $\Box_h = \{m \in M_{\mathbb{R}} \mid \langle m,n\rangle \geq h(n), \forall n \in N_{\mathbb{R}}\}$.

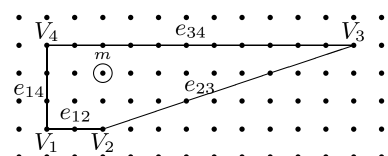
Let V_i be the vertex of $\Box_h$ corresponding to the element of $|D|$ that does not pass through the T_N-invariant point p_i, for $i = 1,\ldots,r$. We denote by $C_{j,k}$ the T_N-invariant curve joining the points p_j and p_k (if it exists), by $\epsilon_{j,k}$ the edge of $\Box_h$ corresponding to the elements of $|D|$ that does not contain $C_{j,k}$ (i.e. the edge joining V_j and V_k), and by $N_{j,k}$ the number of integer points lying on $e_{j,k}$.
In order to explain the notations, we are going to see the example of the system $|3H+2F|$ on the Hirzebruch surface $\mathbb{F}_3$ (we denote by H a rational 3-curve and by F the fiber).

Fan for the $\mathbb{F}_3$ surface



Picture 1

$\Box_h$ for $2F+3H$ on $\mathbb{F}_3$

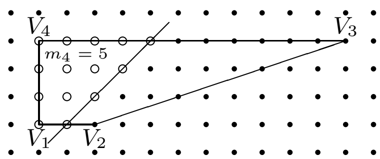


Picture 2

The first picture represents the complete fan of a $\mathbb{F}_3$ surface. In the second picture the integer points of $\Box_h$ represent the elements of $H^0(2F+3H)$. Here for instance we have that $e_{1,4}$ corresponds a fiber F, and $N_{1,4} = 4$.
To each $m \in \Box_h$ there is associated an element $s_m \in H^0(2F+3H)$. To calculate the multiplicity of s_m at p_i we translate the polytope $\Box_h$ in such a way that V_i coincides with the vertex of the dual cone $\sigma_i^\vee$, and hence we express m as a combination of the generators of $\sigma_i^\vee$. The multiplicity of s_m at p_i is just the sum of the coefficients in that combination. For instance, if m is the point in the picture we have that the multiplicities of s_m at $p_1,\ldots,p_4$ are $4,8,7,3$ respectively.

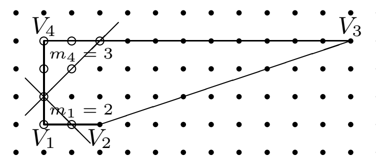
Let us fix the T_N-invariant point p_i and hence the vertex V_i. There exist n edges $e_{i,k_1},\ldots,e_{i,k_n}$, passing through V_i (because the variety X is smooth). Let us denote by H_i the hyperplane meeting the edges e_{i,k_j} $(j = 1,\ldots,n)$ in the m_ith integer point, starting from V_i (we consider the extension of the edge if the number N_{i,k_j} of integer points on it is smaller than m_i). To impose that the system passes through the point p_i with multiplicity m_i is equivalent to cut from $\Box_h$ its intersection with the half-space determined by H_i and containing the vertex V_i. It is clear that the number of integer points we are cutting is exactly $\binom{m_i+n-1}{n}$ (which are the conditions imposed by a point of multiplicity m_i) if and only if the number of integer points lying on each edge e_{i,k_j}, is at least m_i (i.e. $N_{i,k_j} \geq m_i$), for $j = 1,\ldots,n$.

$| \, 2F + 3H - 5p_4 \, |$ on $\mathbb{F}_3$



Picture 3

$| \, 2F + 3H - 2p_1 - 3p_4 \, |$ on $\mathbb{F}_3$



Picture 4

In Picture 3 and 4 we represent the special systems $| \, 2F + 3H - 5p_4 \, |$ and $| \, 2F + 3H - 2p_1 - 3p_4 \, |$ on $\mathbb{F}_3$. The speciality of the first system is due to the conditions imposed on p_4, while the speciality of the second one is due to the overlapping of conditions imposed on p_1 and p_4.

Notation 1.1. Throughout the paper, by abuse of notation, we will use the same symbol to designate a linear system and the corresponding sheaf.

Definition 1.2. We say that a linear system $\mathcal{L}$ on a smooth variety X is *special* if $h^1(X, \mathcal{L}) \neq 0$.

Given an ample, non-special divisor D on X, (i.e. such that $\mathcal{O}_X(D)$ is non-special), r points $p_1, \ldots, p_r \in X$, and r non negative integers $m_1, \ldots, m_r$, we denote by $\mathcal{L}(D, m_1, \ldots, m_r)$ the linear system of divisors linearly equivalent to D and passing through p_i with multiplicity m_i. Therefore, if we denote by Z the 0-dimensional scheme of multiple points, the linear system corresponds to the sheaf $\mathcal{O}_X(D) \otimes \mathcal{I}_Z$.

Definition 1.3. The *virtual dimension* of the system $\mathcal{L}(D, m_1, \ldots, m_r)$, is denoted by $v(\mathcal{L}(D, m_1, \ldots, m_r))$, and is defined to be the difference between the dimension of the complete system $|D|$ and the conditions imposed by the multiple points i.e.

$$v(\mathcal{L}(D, m_1, \ldots, m_r)) = h^0(X, \mathcal{O}_X(D)) - h^0(Z, \mathcal{O}_Z(D)) - 1.$$

From the exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0,$$

tensoring with $\mathcal{O}_X(D)$ and taking cohomology, we obtain

$$0 \to \mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(X, \mathcal{O}_X(D)) \to \mathrm{H}^0(Z, \mathcal{O}_Z \otimes \mathcal{O}_X(D)) \to \mathrm{H}^1(X, \mathcal{L}) \to 0,$$

where, for simplicity of notation, we put $\mathcal{L} := \mathcal{L}(D, m_1, \ldots, m_r)$. Therefore

$$v(\mathcal{L}) = h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) - 1,$$

and, since the effective dimension of $\mathcal{L}$ is $h^0(X, \mathcal{L}) - 1$, the system is special if and only if its virtual dimension is smaller than the effective one.

2. Main Theorem and Conjecture

Let X be a smooth n-dimensional toric variety, and let D be an ample divisor on it. We fix r points $p_1, p_2, \ldots, p_r$ on X, T_N-invariant, r non negative integers $m_1, m_2, \ldots, m_r$, and we consider the linear system $\mathcal{L}(D, m_1, \ldots, m_r)$.
Let C be a curve on X, passing through some of the points p_i, say $p_1, \ldots, p_s$, $s \leq r$. We define the intersection of the system $\mathcal{L}(D, m_1, \ldots, m_r)$ and the curve C as the intersection product of their strict transforms $\mathcal{L}'$ and C' on X', the blow up of X along the points $p_1, \ldots, p_s$. Therefore, if we denote by $E_i \cong \mathbb{P}^{n-1}$ the

exceptional divisor corresponding to the point p_i, and by $C' = \overline{\pi^{-1}(C \setminus \{p_1 \dots p_r\})}$, since $C' \cdot E_i = 1$ for each $i = 1 \dots r$, then we have the following formula:

$$\begin{aligned} C \cdot \mathcal{L}(D, m_1, \dots, m_r) &= C' \cdot \mathcal{L}' \\ &= C' \cdot (\pi^* D - \textstyle\sum_{i=1}^r m_i E_i) \\ &= C \cdot D - \textstyle\sum_{i=1}^s m_i. \end{aligned} \tag{2.1}$$

In order to simplify the proof of our main theorem we are going to state and prove two lemmas.

Lemma 2.1. *The system $\mathcal{L}(D, m_1, \dots, m_r)$ is special if and only if there exist two points, say p_1 and p_2, such that the system $\mathcal{L}(D, m_1, m_2)$ is special.*

Proof. The system $\mathcal{L}(D, m_1, \dots, m_r)$ is special if and only if one of the following hold.
On one hand, there can exist a point, say p_1, such that m_1 is bigger than N_{1,k_j} for some index $j \in \{1, \dots, n\}$, which means that the system $\mathcal{L}(D, m_1)$ (and hence also $\mathcal{L}(D, m_1, m_2)$) is already special.
On the other hand, we have speciality if some of the half-spaces we are cutting have a nonempty intersection inside the polytope $\square_h$. If this is the case, in particular there must exist two of the half-spaces, corresponding to say H_1 and H_2, intersecting inside $\square_h$, which is equivalent to say that the system $\mathcal{L}(D, m_1, m_2)$ is already special. $\square$

Lemma 2.2. *Let D and $C_{j,k}$ be as above. Then the following equality holds:*

$$D \cdot C_{j,k} = N_{j,k} - 1. \tag{2.2}$$

Proof. Let us write $D = \sum_{i=1}^m \alpha_i D_i$, where D_i is the T_N-invariant divisor on X, corresponding to the 1-cone v_i, and let us take the curve $C_{j,k}$, corresponding to the $(n-1)$-cone $\tau = \langle v_1, \dots, v_{n-1} \rangle$. Let us denote by v_n and v_{n+1} the 1-cones which complete τ to the two n-cones containing it. Therefore the intersection product of $C_{j,k}$ with D depends only on its intersection with the D_i, for $i = 1, \dots, n+1$ (see [7], Chapter 2). Actually we have

$$D \cdot C_{j,k} = \sum_{i=1}^{n+1} \alpha_i D_i \cdot C_{j,k}.$$

Since $\langle v_1, \dots, v_{n-1}, v_n \rangle$ and $\langle v_1, \dots, v_{n-1}, v_{n+1} \rangle$ are two n-cones, it follows that $D_n \cdot C_{j,k} = D_{n+1} \cdot C_{j,k} = 1$. Let us then calculate $D_i \cdot C_{j,k}$, for $i = 1, \dots, n-1$.
Let us remark that, via unimodular transformation, we can always suppose that $v_i = e_i$, for $i = 1, \dots, n$, (i.e. the canonical base for $\mathbb{R}^n$. Since $|v_1, \dots, v_{n-1}, v_{n+1}| = -|v_1, \dots, v_{n-1}, v_n| = -1$, the last coordinate of v_{n+1} must be -1 and hence we can write $v_{n+1} = (-\gamma_1, \dots, -\gamma_{n-1}, -1)$. Therefore the following equality holds

$$v_n + v_{n+1} + \gamma_1 v_1 + \dots + \gamma_{n-1} v_{n-1} = 0,$$

which gives (see [7]) $D_i \cdot C_{j,k} = \gamma_i$, for $i = 1, \dots, n-1$, and hence

$$D \cdot C_{j,k} = \sum_{i=1}^{n-1} \alpha_i \gamma_i + \alpha_n + \alpha_{n+1}. \tag{2.3}$$

Let us computate now $N_{j,k}$, i.e. the number of integer points on the edge $e_{j,k}$ of $\square_h$. This edge lies on the line $r = \{x_i = -\alpha_i,\ i = 1, \dots n-1\}$. In particular

it is parallel to the x_n-axis and hence $N_{j,k} = l + 1$, where l is the lenght of $e_{j,k}$. Therefore, since the edge is cut by the two hyperplanes

$$\begin{aligned} H &= \{w \mid \langle w, v_n \rangle = -\alpha_n\} \\ &= \{x_n = -\alpha_n\} \end{aligned}$$

and

$$\begin{aligned} H' &= \{w' \mid \langle w', v_{n+1} \rangle = -\alpha_{n+1}\} \\ &= \{\gamma_1 x_1 + \dots \gamma_{n-1} x_{n-1} + x_n = \alpha_{n+1}\}, \end{aligned}$$

its endpoints are $p = (-\alpha_1, \dots, -\alpha_{n-1}, -\alpha_n)$ and $p' = (-\alpha_1, \dots, -\alpha_{n-1}, \sum_{i=1}^{n-1} \gamma_i \alpha_i + \alpha_{n+1})$. Therefore, recalling (2.3),

$$\begin{aligned} N_{j,k} - 1 &= l \\ &= \textstyle\sum_{i=1}^{n-1} \gamma_i \alpha_i + \alpha_{n+1} + \alpha_n \\ &= D \cdot C_{j,k}. \end{aligned}$$

$\square$

Main Theorem. *The linear system $\mathcal{L}(D, m_1, \dots, m_r)$ is special if and only if there exists a T_N-invariant curve C passing through two of the p_i's, such that $\mathcal{L}(D, m_1, \dots, m_r) \cdot C \leq -2$.*

Proof. Because of Lemma 2.1 we can fix our attention on the system $\mathcal{L}(D, m_1, m_2)$, relating its speciality with the number $N_{1,2}$.
In fact, if the speciality of our system is due to only one point, say p_1, then there exists an edge, say $e_{1,2}$ such that $m_1 \geq N_{1,2} + 1$, as we have seen in the proof of previous Lemma 2.1. On the other hand, if the speciality is due to both p_1 and p_2, then the two half-spaces determined by H_1 and H_2 intersect inside $\square_h$. In this case their intersection contains at least one point of the edge $e_{1,2}$.
Therefore, in both cases we have that the system is special if and only if the following equality holds:

$$m_1 + m_2 \geq N_{1,2} + 1. \tag{2.4}$$

From the intersection formula (2.1) and from Lemma 2.2,

$$\begin{aligned} \mathcal{L}(D, m_1, m_2) \cdot C_{1,2} &= D \cdot C_{1,2} - (m_1 + m_2) \\ &= N_{1,2} - 1 - (m_1 + m_2). \end{aligned}$$

and hence, from (2.4), the system is special if and only if $\mathcal{L}(D, m_1, m_2) \cdot C_{1,2} \leq -2$. $\square$

Example 2.3. Let us consider the Hirzebruch surface $\mathbb{F}_3$. This is the toric surface associated to the complete two-dimensional fan of Picture 1. There exist 4 invariant 1-dimensional varieties, namely the (-3)-curve E, one rational 3-curve H and two fibers F_1 and F_2. Their intersections give rise to 4 invariant points, which we denote by p_i, $i = 1, \dots, 4$ as in Picture 2.
Let us consider now the linear system $|3H + 2F - 5p_4|$. Blowing up along p_4 one obtains the system $|3\pi^* H + 2\pi^* F - 5E_4|$, where E_4 is the exceptional divisor of the blowing up π. Now observe that $|\pi^* F - E_4|$ is a curve of genus 0 and that $(3\pi^* H + 2\pi^* F - 5E_4) \cdot (\pi^* F - E_4) = -2$, hence by the remark below, the system must be special (as we can see geometrically from Picture 3). In a similar way also the system $|3H + 2F - 2p_1 - 3p_4|$ is special since, blowing up p_1 and p_4 one has $(3\pi^* H + 2\pi^* F - 2E_1 - 3E_4) \cdot (\pi^* F - E_1 - E_4) = -2$ where $|\pi^* F - E_1 - E_4|$ is the fiber through p_1, p_4 (see Picture 4).

Remark 2.4. Observe that if S is a rational surface, D is an ample, non special divisor, and $\mathcal{L}$ is a linear system of curves linearly equivalent to D and such that there exists a rational, irreducible curve $C \subset S$ with $\mathcal{L} \cdot C \leq -2$, then $\mathcal{L}$ is special. In fact, the effective dimension of $\mathcal{L}$ equals that of $\mathcal{L} - C$, and we are going to see that the virtual dimension $v(\mathcal{L})$ is smaller than $v(\mathcal{L} - C)$ (and hence, in particular, $v(\mathcal{L})$ is smaller than the effective dimension of $\mathcal{L}$, and the system is special). We recall that $v(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) - 1$ and, being S rational, $h^2(\mathcal{L}) = 0$, which implies $v(\mathcal{L}) = \chi(\mathcal{L}) - 1$. By Riemann-Roch, $\chi(\mathcal{L}) = (\mathcal{L}^2 - \mathcal{L} \cdot K_S)/2 + 1$. Therefore $v(\mathcal{L}) = (\mathcal{L}^2 - \mathcal{L} \cdot K_S)/2$ and $v(\mathcal{L} - C) = ((\mathcal{L} - C)^2 - (\mathcal{L} - C) \cdot K_S)/2 = v(\mathcal{L}) + g(C) - 1 - \mathcal{L} \cdot C$. The rationality of C implies that

$$v(\mathcal{L}) = v(\mathcal{L} - C) + \mathcal{L} \cdot C + 1.$$

Hence, if $\mathcal{L} \cdot C \leq -2$, we have that $v(\mathcal{L}) < v(\mathcal{L} - C)$.

Consider the Hirzebruch surface $\mathbb{F}_6$ and the linear system $\mathcal{L}(\mathcal{O}_{\mathbb{F}_6}(0,4), 3^{11})$. The virtual dimension of this system is -2. Let C be an element of $\mathcal{L}(\mathcal{O}_{\mathbb{F}_6}(2,1), 1^{11})$, then C is rational and $\mathcal{L} \cdot C = -1$. Let $\Gamma_6 \in \mathcal{L}(\mathcal{O}_{\mathbb{F}_6}(-6,1))$ be the (-6)-curve. Then we have the following:

$$\begin{array}{rcl} \mathcal{L} \cdot C & = & -1 \\ (\mathcal{L} - C) \cdot \Gamma_6 & = & -2 \\ (\mathcal{L} - C - \Gamma_6) \cdot C & = & -2 \end{array}$$

Now, since $3C + \Gamma_6 = 3\mathcal{L}(\mathcal{O}_{\mathbb{F}_6}(2,1), 1^{11}) + \mathcal{L}(\mathcal{O}_{\mathbb{F}_6}(-6,1)) = \mathcal{L}(\mathcal{O}_{\mathbb{F}_6}(0,4), 3^{11})$, then the initial system is not empty and therefore special. Observe that there exists no curve E on $\mathbb{F}_6$ such that $\mathcal{L} \cdot E \leq -2$.

This motivates the following procedure in the case of a smooth rational surface X.

Step 1 - Let $\mathcal{L}$ be a non-empty linear system on a rational surface X. If there exists a rational curve C on X such that $\mathcal{L} \cdot C \leq -1$, replace $\mathcal{L}$ with $\mathcal{L} - C$ and restart from Step 1.
Step 2 - If the last system has virtual dimension bigger than that of the initial one then we say that $\mathcal{L}$ is a (-1)-*special system*.

Considering the remark above and our main theorem, we conjecture that:

Conjecture 2.5. *Let D be an ample, non-special divisor on a rational surface S, and let $p_1, \ldots, p_r$ be r points on S. Then the linear system $\mathcal{L} = \mathcal{L}(D, m_1, \ldots, m_r)$ is special if and only if it is (-1)-special.*

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